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Bifurcations of steady states in a class of lattices of nonlinear discrete Klein–Gordon type with double-quadratic on-site potential

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Abstract
By using the method of symbolic dynamics, we study the bifurcations of steady states in a class of lattices of nonlinear discrete Klein–Gordon type with double-quadratic on-site potential. We derive by virtue of the admissible condition the critical value ε_0 of the coupling strength, below which the steady states persist without bifurcations. If the coupling coefficient ε passes through the critical value, some of the steady states disappear. Meanwhile there are no new steady states created as ε varies. We obtain bifurcation values of some lower-order spatially periodic steady states by introducing the concept 'characteristic polynomial' of periodic sequences.

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1. Introduction

In this paper, we study a class of one-dimensional lattices of nonlinear discrete Klein–Gordon type with Hamiltonian

$$H = \sum_n \left[\frac{p_n^2}{2} + \frac{\varepsilon}{2}(x_n - x_{n-1})^2 + V(x_n) \right]$$

where $V(u)$ is a symmetric, double-quadratic potential [5, 15]:

$$V(u) = \tfrac{1}{2}(u - \operatorname{sgn}(u))^2$$

$\varepsilon > 0$ measures the coupling strength. The coupled ordinary differential equations defined by this Hamiltonian are

$$\ddot{x}_n = \varepsilon(x_{n+1} - 2x_n + x_{n-1}) + f(x_n) \tag{1.1}$$

where $f(u) = -u + \operatorname{sgn}(u)$.

Recently, much work has been devoted to studying nonlinear localized solutions in lattice systems, e.g., discrete breathers (DBs) [5, 12, 13, 15], which are time periodic and spatially localized, and static localized solutions [3, 4, 9, 12, 14]. The existence and properties of nonlinear localized solutions of system (1.1) have been studied in [5, 10, 15–18]. To investigate the existence of discrete breathers and multibreathers, one should know, first of all, the steady states $\{v_n\}$ of system (1.1), then write the solution $x_n(t)$ as

$$x_n(t) = v_n + \phi_n(t). \tag{1.2}$$

Substituting (1.2) into (1.1), one obtains the equations ϕ_n should satisfy. Then one looks for the conditions on frequencies to ensure that $\phi_n(t)$ are time periodic and that the amplitudes of $\phi_n(t)$ are exponentially decaying (see [15]). Here we see a close relation between discrete breathers and steady states in system (1.1).

In this paper we study the bifurcations of steady states of system (1.1). There are several reasons. First, as introduced above, there is a close relation between DBs and the localized steady states. Second, the existence and stability of steady states in lattice systems also attract much attention in the literature [3, 9, 12, 14]. Third, there is a simple and efficient method called 'anti-integrability' [2, 14] for investigating steady states in weakly coupled systems, but this method cannot be applied directly to system (1.1) because of the discontinuity of the nonlinear term f. When one applies the 'anti-integrability' method, one obtains a sufficient condition, i.e., there exists $\varepsilon_0 > 0$ such that all the steady states of system (1.1) in the limit of vanishing coupling persist for $0 < \varepsilon < \varepsilon_0$. One needs to know whether these steady states persist or bifurcate for $\varepsilon > \varepsilon_0$, and whether there are new steady states created as ε varies. Finally, when bifurcations actually occur, which steady state undergoes bifurcation and when?

We try to answer some of these questions for system (1.1). In fact, many interesting results have been obtained on the persistence and bifurcations of the steady states for system (1.1). The same model, even with asymmetric on-site potential, was studied by Schilling [17], Reichert and Schilling [16], Häner and Schilling [6] and Vollmer *et al* [18]. In [16, 17], by the use of a Green function method, it was rigorously proved that for $\varepsilon < \frac{3}{4}$ (the critical value $\varepsilon_0 = \frac{3}{4}$ corresponds to the critical value $\eta_c = \frac{1}{3}$ in [16, 17]) there is a steady state corresponding to each sequence $\theta = (\theta_n) \in \{-1, 1\}^Z$. The remaining question is whether the steady state is unique. Here we construct 'Milnor's map' [8], and apply the contraction mapping theorem to show that the steady state corresponding to each sequence is unique if it exists. Hence we have established a one-to-one correspondence between the steady states and the sequences of pseudo spins $\theta \in \{-1, 1\}^Z$ provided $\varepsilon < \frac{3}{4}$, that is, there are no bifurcations for the steady states of system (1.1) if $\varepsilon \in \left[0, \frac{3}{4}\right)$. Note that for $\varepsilon = 0$, this one-to-one correspondence is obvious. But it is not trivial that this one-to-one correspondence exists for $\varepsilon > 0$ (see [2]). It involves the problem of structural stability of system (1.1) (see also [12]). The bifurcation behaviour for $\varepsilon > \frac{3}{4}$ (corresponding to $\eta > \frac{1}{3}$ in [18]) was studied in [18] in order to calculate the number of metastable states. Here we introduce the characteristic polynomial of the periodic sequence, and apply Descartes's rule of signs to show that for the periodic sequences with periods less than or equal to 5, the corresponding steady states disappear forever when ϵ passes through the bifurcation values. This partly supports the observation in [18] that the numbers of metastable configurations decrease as ϵ increases.

2. Persistence and bifurcations of steady states

The steady states of system (1.1) were studied in [16, 17] with a Green function method. The correspondence between the steady states of system (1.1) and a two-dimensional map was also established in [16]. In this section, we prove by the contraction mapping theorem that the steady state corresponding to each sequence $\theta = (\theta_n)$ is unique, which can guarantee the one-to-one correspondence between the steady states and all the sequences provided $\varepsilon < \frac{3}{4}$. This implies that there are no new-born steady states by adding the spatial interactions.

The steady states of system (1.1) can be transformed into the zeros of map $\mathcal{F}_\varepsilon$ defined in ℓ^∞:

$$(\mathcal{F}_\varepsilon(x))_n = f(x_n) + \varepsilon(x_{n+1} - 2x_n + x_{n-1}).$$

For each element $x = (x_n) \in \ell^\infty$, we introduce a sequence

$$\theta = (\theta_n) \in \Sigma_2 = \{-1, 1\}^Z$$

given by $\theta_n = \mathrm{sgn}(x_n)$. Then the zero $x = (x_n)$ of $\mathcal{F}_\varepsilon$ satisfies

$$\theta_n - x_n + \varepsilon(x_{n+1} - 2x_n + x_{n-1}) = 0$$

which leads to

$$x_n = \frac{\varepsilon(x_{n+1} + x_{n-1}) + \theta_n}{1 + 2\varepsilon}.$$

On the other hand, for each sequence $\theta = (\theta_n)$, define a map M_θ from ℓ^∞ to ℓ^∞:

$$(M_\theta(x))_n = \frac{\varepsilon(x_{n+1} + x_{n-1}) + \theta_n}{1 + 2\varepsilon}.$$

M_θ is called Milnor's map in [8]. One can easily check that M_θ is a contraction for $\varepsilon > 0$. Indeed, for $x = (x_n)$ and $y = (y_n)$ in ℓ^∞,

$$\begin{aligned}\|M_\theta(x) - M_\theta(y)\| &= \sup_{n\in Z}\left|\frac{\varepsilon}{1+2\varepsilon}(x_{n+1} - y_{n+1} + x_{n-1} - y_{n-1})\right| \\ &\leqslant \frac{2\varepsilon}{1+2\varepsilon}\|x - y\| = \delta\|x - y\|\end{aligned}$$

in which $\delta < 1$ for $\varepsilon > 0$. By the contraction mapping theorem, M_θ has a unique fixed point $x = (x_n)$. If the fixed point $x = (x_n)$ of M_θ satisfies the admissible condition

$$\theta_n = \mathrm{sgn}(x_n) = \begin{cases} -1 & x_n \leqslant 0 \\ 1 & x_n > 0 \end{cases}$$

then x is a zero of $\mathcal{F}_\varepsilon$.

Let

$$\mathcal{A} = \{x = (x_n) | x \text{ is the fixed point of } M_\theta, \theta = (\theta_n) \in \Sigma_2 \text{ and } \theta_n = \mathrm{sgn}(x_n)\}.$$

Then $\mathcal{A}$ contains all the zeros of $\mathcal{F}_\varepsilon$. Indeed, if $\mathcal{F}_\varepsilon$ has a zero $x = (x_n)$, then it is exactly the unique fixed point of M_θ, in which $\theta = (\theta_n)$ and $\theta_n = \mathrm{sgn}(x_n)$.

We refer to $\theta = (\theta_n)$ as admissible if $\mathcal{F}_\varepsilon$ has a zero $x = (x_n)$ with $\theta_n = \mathrm{sgn}(x_n)$, otherwise it is called forbidden.

In the following, we find, for $\varepsilon > 0$, all the zeros of map $\mathcal{F}_\varepsilon$. To this end, we transform the second-order difference equation

$$f(x_n) + \varepsilon(x_{n+1} + x_{n-1} - 2x_n) = 0 \tag{2.1}$$

into a 2D map T:

$$\begin{cases} u_{n+1} = v_n \\ v_{n+1} = -u_n + 2v_n - \frac{1}{\varepsilon} f(v_n) \end{cases}$$

where $(u_n, v_n)^T = (x_{n-1}, x_n)^T$. Then the bounded orbits $\{(u_n, v_n)^T\}$ correspond to the zeros of $\mathcal{F}_\varepsilon$ in ℓ^∞. Let $\theta_n = \mathrm{sgn}(v_n)$. Then a bounded orbit of map T satisfies

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2 + 1/\varepsilon \end{pmatrix}\begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} 0 \\ -\theta_n/\varepsilon \end{pmatrix}. \tag{2.2}$$

Conversely, if $\{(u_n, v_n)^T\}$ satisfies (2.2) and $\theta_n = \mathrm{sgn}(v_n)$, then $\{(u_n, v_n)^T\}$ is a bounded orbit of T. Let

$$U = \begin{pmatrix} u \\ v \end{pmatrix} \qquad \sigma_n = \begin{pmatrix} 0 \\ -\theta_n/\varepsilon \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} 0 & 1 \\ -1 & 2+2k \end{pmatrix}$$

where $2k = 1/\varepsilon > 0$. Then

$$U_{n+1} = AU_n + \sigma_n.$$

The eigenvalues of A are $\lambda_1 = 1 + k + \sqrt{2k + k^2} > 1$, and $\lambda_2 = 1/\lambda_1$. The eigenvectors corresponding to λ_1 and λ_2 are $\xi_1 = (\lambda_2, 1)^T$ and $\xi_2 = (\lambda_1, 1)^T$, respectively. Let $P = (\xi_1, \xi_2)$. Then

$$P^{-1} = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} 1 & -\lambda_1 \\ -1 & \lambda_2 \end{pmatrix}.$$

Under the transformation $Z = P^{-1}U$, we have

$$Z_{n+1} = P^{-1}U_{n+1} = P^{-1}APZ_n + P^{-1}\sigma_n = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} Z_n + P^{-1}\sigma_n.$$

Denoting the components of Z by z^1 and z^2, we have

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \lambda_2 & \lambda_1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}$$

which leads to $v = z^1 + z^2$. Consequently, given $\theta = (\theta_n)$, if $\mathrm{sgn}\big(z_n^1 + z_n^2\big) = \theta_n$, then $\{Z_n\}$ is a bounded orbit, that is, $\{U_n\} = \{PZ_n\}$ is a bounded orbit of map T.

Let

$$\alpha = \frac{-\lambda_1}{(\lambda_1 - \lambda_2)\varepsilon} \qquad \beta = \frac{\lambda_2}{(\lambda_1 - \lambda_2)\varepsilon}.$$

Then $P^{-1}\sigma_n = (\alpha\theta_n, \beta\theta_n)^T$. In the following, we denote $\lambda_2 = 1/\lambda_1$ by τ satisfying $0 < \tau < 1$ for $0 < \varepsilon < +\infty$. From the fact

$$Z_{n+1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} Z_n + \begin{pmatrix} \alpha\theta_n \\ \beta\theta_n \end{pmatrix}$$

it follows that $z_{n+1}^1 = \lambda_1 z_n^1 + \alpha\theta_n$. Consequently,

$$z_n^1 = -\alpha\tau \sum_{k=0}^{\infty} \tau^k \theta_{n+k}.$$

By a similar calculation we deduce that

$$z_n^2 = \beta \sum_{k=1}^{\infty} \tau^{k-1} \theta_{n-k}.$$

Hence

$$z_n^1 + z_n^2 = \frac{1}{(\lambda_1 - \lambda_2)\varepsilon} \left[\theta_n + \sum_{k=1}^{\infty} \tau^k (\theta_{n+k} + \theta_{n-k}) \right].$$

Then the necessary and sufficient condition for $\{Z_n\}$ being a bounded orbit is

$$\begin{cases} \theta_n + \sum_{k=1}^{\infty} \tau^k (\theta_{n+k} + \theta_{n-k}) \leqslant 0 & \text{if} \quad \theta_n = -1 \\ \theta_n + \sum_{k=1}^{\infty} \tau^k (\theta_{n+k} + \theta_{n-k}) > 0 & \text{if} \quad \theta_n = 1. \end{cases} \tag{2.3}$$

For $\theta = (\theta_n) \in \Sigma_2$, let

$$w_n(\theta) = \sum_{k=1}^{\infty} \tau^k (\theta_{n+k} + \theta_{n-k}).$$

Then condition (2.3) is equivalent to

$$\begin{array}{lllll} w_n(\theta) \leqslant 1 & \text{for} & n \in \mathbb{Z} & \text{if} & \theta_n = -1 \\ w_n(\theta) > -1 & \text{for} & n \in \mathbb{Z} & \text{if} & \theta_n = 1. \end{array}$$

Let

$$\xi(\theta) = \inf_{\theta_n = 1} w_n(\theta) \qquad \text{and} \qquad \eta(\theta) = \sup_{\theta_n = -1} w_n(\theta).$$

For $\theta = (\theta_n)$, from the above discussion, it follows that if $\xi(\theta) > -1$ and $\eta(\theta) \leqslant 1$, then θ is admissible, i.e., there exists a unique bounded orbit $\{U_n\}$ corresponding to it, or coming back to our original problem, there is a unique zero $x = (x_n)$ of $\mathcal{F}_\varepsilon$ corresponding to θ.

In summary, for $\theta = (\theta_n) \in \Sigma_2$ satisfying $\xi(\theta) > -1$ and $\eta(\theta) \leqslant 1$, the unique zero $x = (x_n)$ of $\mathcal{F}_\varepsilon$ corresponding to θ is

$$x_n = \frac{1}{(\lambda_1 - \lambda_2)\varepsilon} \left[\theta_n + \sum_{k=1}^{\infty} \tau^k (\theta_{n+k} + \theta_{n-k}) \right].$$

In what follows, the symbols +1 and -1 are abbreviated to + and $-$, respectively, and the superscript ∞ represents repetition.

For each sequence $\theta = (\theta_n)$, we have

$$\xi(\theta) \geqslant \xi((-^{\infty} + -^{\infty})) = -2(\tau + \tau^2 + \cdots) = \frac{-2\tau}{1-\tau}$$

and

$$\eta(\theta) \leqslant \eta((+^{\infty} - +^{\infty})) = 2(\tau + \tau^2 + \cdots) = \frac{2\tau}{1-\tau}.$$

Theorem 2.1. *If the coupling coefficient $\varepsilon < 3/4$, then corresponding to each sequence $\theta = (\theta_n)$, there is a unique zero of $\mathcal{F}_\varepsilon$.*

Proof. Simple calculation shows that $\tau < 1/3$ for $\varepsilon < 3/4$, which implies $\xi(\theta) > -1$ and $\eta(\theta) \leqslant 1$ for any $\theta \in \Sigma_2$. □

So far we have shown that all the zeros of $\mathcal{F}_0$ can be continued to $\varepsilon < 3/4$ with no new-born zeros. $\varepsilon_0 = 3/4$ is the uniform bound for $\theta \in \Sigma_2$. We should mention here that, except for the uniqueness, the result in theorem 2.1 was first obtained in [16, 17]. The critical value $\tau = \frac{1}{3}$ corresponds to $\eta = \frac{1}{3}$ in [16, 17]. However, whether $\varepsilon = \frac{3}{4}$ is a bifurcation value needs further analysis. That is, one needs to show that there always exist forbidden sequences whenever $\varepsilon > \frac{3}{4}$. The following theorem shows that not all of these zeros at $\varepsilon = 0$ can be uniformly continued further than ε_0, that is, bifurcations occur as ε passes through $\varepsilon_0 = \frac{3}{4}$.

Let us denote by $[-^m + -^m]$ the sequences containing the string $(-^m + -^m)$ with other symbols being arbitrary.

Theorem 2.2. *For any $\varepsilon > 3/4$, there exists integer m, such that the sequences $[-^m + -^m]$ and $[+^m - +^m]$ are forbidden.*

Proof.

$$\xi([-^m + -^m]) \leqslant \xi((+^{\infty} -^m + -^m +^{\infty})) \leqslant \frac{-2\tau}{1-\tau} + \frac{4\tau^{m+1}}{1-\tau} = \frac{-2(\tau - 2\tau^{m+1})}{1-\tau}.$$

For any $1/3 < \tau < 1$, take m large enough so that $3\tau - 4\tau^{m+1} \geqslant 1$, which implies $\xi([-^m + -^m]) \leqslant -1$. Hence the sequences $[-^m + -^m]$ are forbidden. Note that

$$\tau = \tau(\varepsilon) = 1 + \frac{1}{2\varepsilon} - \sqrt{\frac{1}{\varepsilon} + \frac{1}{4\varepsilon^2}} \qquad \text{and} \qquad \tau'(\varepsilon) = \frac{1 + 2\varepsilon - \sqrt{1+4\varepsilon}}{2\varepsilon^2\sqrt{1+4\varepsilon}}$$

hence τ is strictly increasing with respect to ε. Therefore, for any $\varepsilon > 3/4$, we can take m large enough so that $3\tau - 4\tau^{m+1} \geqslant 1$. Then the sequences $[-^m + -^m]$ are forbidden. The same conclusion holds also for sequences $[+^m - +^m]$. □

Of course, there exist sequences which do not undergo bifurcations for $\varepsilon > 0$, for example, sequences $(-^\infty)$ and $(+^\infty)$ corresponding to the two fixed points of map T, $(-^\infty+^\infty)$ and $(+^\infty-^\infty)$ which correspond to heteroclinic orbits, and $(+-)^\infty$ which corresponds to a period-2 orbit. These sequences persist for $\varepsilon \in (0, +\infty)$.

Now let us take a look at the bifurcation values of some lower-order periodic sequences. For sequence $(+-+)^\infty$ with period 3, the admissible condition (2.3) yields three inequalities:

$$2(\tau^3 + \tau^6 + \cdots) > -1 \tag{2.4}$$

$$2(\tau + \tau^2 - \tau^3 + \tau^4 + \tau^5 - \tau^6 + \cdots) \leqslant 1 \tag{2.5}$$

and

$$2(\tau^3 + \tau^6 + \cdots) > -1. \tag{2.6}$$

Inequalities (2.4) and (2.6) hold true obviously for $\tau > 0$ and (2.5) is equivalent to

$$\tau^3 - 2\tau^2 - 2\tau + 1 \geqslant 0. \tag{2.7}$$

Therefore, sequence $(+-+)^\infty$ is admissible if and only if (2.7) holds. We call the polynomial

$$f_3(\tau) = \tau^3 - 2\tau^2 - 2\tau + 1$$

the characteristic polynomial of sequence $(+-+)^\infty$. Similarly, sequence $(-+-)^\infty$ has the same characteristic polynomial as $(+-+)^\infty$. In the following we show that there exists τ_0 such that

$$\tau^3 - 2\tau^2 - 2\tau + 1 > 0 \qquad \text{for} \quad 0 < \tau < \tau_0$$

and

$$\tau^3 - 2\tau^2 - 2\tau + 1 < 0 \qquad \text{for} \quad \tau_0 < \tau < 1.$$

Hence the bifurcation value for sequences $(+-+)^\infty$ and $(-+-)^\infty$ is τ_0, i.e., the two sequences are admissible for $0 < \tau < \tau_0$ and forbidden for $\tau_0 < \tau < 1$. Indeed, we have $f_3(0) = 1 > 0$ and $f_3(1) = -2 < 0$. Consequently, f_3 has at least one zero τ_0 in the interval $(0, 1)$. On the other hand, from the signs of the coefficients we know by Descartes's rule of signs that f_3 has at most two positive real roots. Meanwhile, $1/\tau_0$ is also a positive real root of f_3. So f_3 has exactly one root in $(0, 1)$.

We give the characteristic polynomials of some lower-order periodic sequences in table 1.

One can easily check that other sequences with periods 4 and 5 are always admissible for $0 < \tau < 1$. We have observed that these characteristic polynomials are reflexive, hence if τ^* is a real root, so is $1/\tau^*$. On the other hand, from the signs of the coefficients, we deduce that f_4 (or f_5^i, $i = 1, 2, 3$) has at most two positive real roots. Consequently, each of these polynomials has exactly one positive real root in $(0, 1)$. Therefore, for each of the above sequences, there is a bifurcation value $\tau^* \in (0, 1)$. For $0 < \tau < \tau^*$, the corresponding sequence is admissible, and for $\tau^* < \tau < 1$, the sequence is forbidden. The bifurcation value

Table 1. The characteristic polynomials of some lower-order periodic sequences.

Characteristic polynomial	Sequence
$f_4(\tau) = \tau^4 - 2\tau^3 - 2\tau^2 - 2\tau + 1$	$(+ + + -)^\infty$ and $(- - - +)^\infty$
$f_5^1(\tau) = \tau^5 - 2\tau^4 - 2\tau^3 - 2\tau^2 - 2\tau + 1$	$(+ + + + -)^\infty$ and $(- - - - +)^\infty$
$f_5^2(\tau) = \tau^5 - 2\tau^3 - 2\tau^2 + 1$	$(+ + + - -)^\infty$ and $(- - - + +)^\infty$
$f_5^3(\tau) = \tau^5 - 2\tau^4 - 2\tau + 1$	$(+ + - + -)^\infty$ and $(- - + - +)^\infty$

Table 2. The admissible and forbidden intervals of some lower-order periodic sequences including two homoclinic and two heteroclinic sequences.

Period	Sequence	Admissible interval	Forbidden interval
1	$+^\infty$	$(0, +\infty)$	
1	$-^\infty$	$(0, +\infty)$	
2	$(+ -)^\infty$	$(0, +\infty)$	
3	$(+ + -)^\infty$	$(0, 1.0002)$	$(1.0002, +\infty)$
3	$(- - +)^\infty$	$(0, 1.0002)$	$(1.0002, +\infty)$
4	$(+ + + -)^\infty$	$(0, 0.8089)$	$(0.8089, +\infty)$
4	$(- - - +)^\infty$	$(0, 0.8089)$	$(0.8089, +\infty)$
5	$(+ + + + -)^\infty$	$(0, 0.7676)$	$(0.7676, +\infty)$
5	$(- - - - +)^\infty$	$(0, 0.7676)$	$(0.7676, +\infty)$
5	$(+ + + - -)^\infty$	$(0, 3.3030)$	$(3.3030, +\infty)$
5	$(- - - + +)^\infty$	$(0, 3.3030)$	$(3.3030, +\infty)$
5	$(+ + - + -)^\infty$	$(0, 1.6179)$	$(1.6179, +\infty)$
5	$(- - + - +)^\infty$	$(0, 1.6179)$	$(1.6179, +\infty)$
	$-^\infty + -^\infty$	$(0, 0.75)$	$(0.75, +\infty)$
	$+^\infty - +^\infty$	$(0, 0.75)$	$(0.75, +\infty)$
	$+^\infty -^\infty$	$(0, +\infty)$	
	$-^\infty +^\infty$	$(0, +\infty)$	

differs from sequence to sequence. For example, the bifurcation values of the three period-5 sequences $(+ + + + -)^\infty$, $(+ + + - -)^\infty$ and $(+ + - + -)^\infty$ are 0.3372, 0.5807 and 0.4643, respectively.

We list in table 2 the admissible and forbidden intervals of ε for all the periodic orbits with periods less than or equal to 5, two homoclinic orbits and two heteroclinic orbits.

Next, we give the geometrical explanation of the bifurcation point $\varepsilon = 3/4$ and the persistence of two heteroclinic orbits $(-^\infty +^\infty)$ and $(+^\infty -^\infty)$ for $0 < \tau < 1$, i.e., for $0 < \varepsilon < +\infty$. From the geometrical description we see that the steady states corresponding to the sequences $(-^\infty + \cdots + -^\infty)$ and $(+^\infty - \cdots - +^\infty)$ are spatially localized.

Note that for the linear transformation

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 2 + 2k \end{pmatrix}$$

the unstable and stable manifolds of the origin are two lines with slopes $1/\tau$ and τ, respectively. The two-dimensional map T induced from the second-order difference equation (2.1) is

$$T \begin{pmatrix} u_n \\ v_n \end{pmatrix} = A \begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} 0 \\ 2k \end{pmatrix} \qquad \text{if} \quad v_n \leqslant 0$$

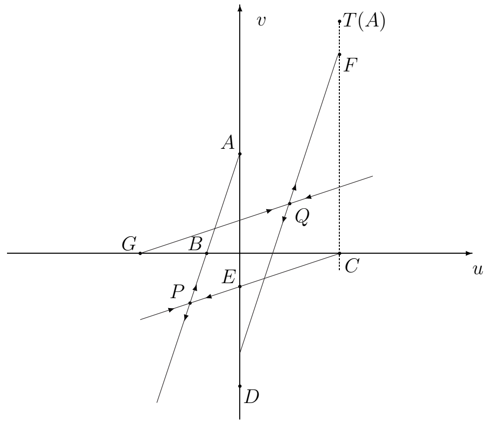

Figure 1. $P = (-1, -1)$ and $Q = (1, 1)$ are the fixed points of T. $PA \subset W^u(P)$, $PC \subset W^s(P)$, $QG \subset W^s(Q)$ and $QF \subset W^u(Q)$. Note that the intersection of PA and QG is always non-empty for $0 < \tau < 1$.

and

$$T\begin{pmatrix} u_n \\ v_n \end{pmatrix} = A\begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} 0 \\ -2k \end{pmatrix} \qquad \text{if} \quad v_n > 0.$$

Let $U = (u, v)^T$, and $a = (1, 1)^T$. If $v - 1 \leqslant 0$, then

$$T(U - a) = AU - Aa + (0, 2k)^T = AU - a.$$

If the v-coordinate of $AU - a$ is less than or equal to zero, then

$$T^2(U - a) = A^2U - a.$$

Hence if the v-coordinate of $A^{m-1}U - a$ is less than or equal to zero, then

$$T^m(U - a) + a = A^mU \qquad m = 1, 2, \ldots.$$

Similarly, if the v-coordinate of $A^{m-1}U + a$ is greater than zero, then

$$T^m(U + a) - a = A^mU \qquad m = 1, 2, \ldots.$$

From these facts, we depict part of the stable and unstable manifolds of the fixed points $P = (-1, -1)$ and $Q = (1, 1)$, respectively, see figure 1.

From the figure we know that for $0 < \tau < 1$,

$$W^u(P) \cap W^s(Q) \neq \emptyset \qquad \text{and} \qquad W^s(P) \cap W^u(Q) \neq \emptyset$$

which implies that two heteroclinic orbits $(-^{\infty}+^{\infty})$ and $(+^{\infty}-^{\infty})$ persist for $\tau \in (0, 1)$.

Let A and B be the intersection points of $W^u(P)$ and v-axis, u-axis respectively, and C be the intersection point of $W^s(P)$ and u-axis. Then

$$A = (0, 1/\tau - 1) \qquad B = (\tau - 1, 0) \qquad \text{and} \qquad C = (1/\tau - 1, 0).$$

Note that

$$F = (1/\tau - 1, 1 + 1/\tau^2 - 2/\tau)$$

is the intersection point of $W^u(Q)$ and the line $u = 1/\tau - 1$ passing through point C. Let

$$D = (0, 1/\tau - 1 - 4k).$$

Denote by $\overset{\circ}{B}A$ the segment BA excluding point B. Then $T(\overset{\circ}{B}A) = \overset{\circ}{D}T(A)$ in which

$$T(A) = (1/\tau - 1, 1/\tau^2 - 2/\tau - 2\tau + 3).$$

One can easily check that $T(A)$ lies above point F. If D lies below $E = (0, \tau - 1)$, then $\overset{\circ}{D}T(A) \cap W^s(P) \neq \emptyset$, which implies that the homoclinic orbit $(-^{\infty} + -^{\infty})$ appears and then all the sequences are admissible. It is easy to check that D laying below E is equivalent to $\varepsilon < 3/4$.

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